

# The Cauchy Problem on the Plane for the Dispersionless Kadomtsev - Petviashvili Equation

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## Abstract

We construct the formal solution of the Cauchy problem for the dispersionless Kadomtsev - Petviashvili equation as application of the Inverse Scattering Transform for the vector field corresponding to a Newtonian particle in a time-dependent potential. This is in full analogy with the Cauchy problem for the Kadomtsev - Petviashvili equation, associated with the Inverse Scattering Transform of the time dependent Schrödinger operator for a quantum particle in a time-dependent potential.

1. Dispersionless (or quasi-classical) limits of integrable partial differential equations (PDEs) arise in various problems of Mathematical Physics and are intensively studied in the recent literature (see, f.i., [1, 2, 3, 4, 5]). In particular, a quasi-classical dressing has been developed [4] for the prototypical example of the dispersionless Kadomtsev - Petviashvili (dKP) (or Khokhlov-Zabolotskaya) equation:

$$u_{tx} + u_{yy} + (uu_x)_x = 0, \quad u = u(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R}. \quad (1)$$

In this paper we construct the formal solution of the Cauchy problem on the plane for the following system of PDEs in 2+1 dimensions:

$$\begin{aligned} u_{xt} + u_{yy} &= -(uu_x)_x - v_x u_{xy} + v_y u_{xx}, & u, v \in \mathbb{R}, & \quad x, y, t \in \mathbb{R}, \\ v_{xt} + v_{yy} &= -uv_{xx} - v_x v_{xy} + v_y v_{xx} \end{aligned} \quad (2)$$

and for its  $v = 0$  reduction, the dKP equation (1), as application of the recently developed Inverse Scattering Transform (IST) for vector fields [6].

Indeed the system (2) arises as the compatibility condition of the Lax pair

$$\hat{L}_1\psi = 0, \quad \hat{L}_2\psi = 0, \quad (3)$$

implying  $[\hat{L}_1, \hat{L}_2] = 0$ , where  $\hat{L}_1, \hat{L}_2$  are the following vector fields:

$$\begin{aligned} \hat{L}_1 &\equiv \partial_y + (p + v_x)\partial_x - u_x\partial_p, \\ \hat{L}_2 &\equiv \partial_t + (p^2 + pv_x + u - v_y)\partial_x + (-pu_x + u_y)\partial_p. \end{aligned} \quad (4)$$

Setting  $v = 0$  in (4), one obtains the Lax pair of the dKP equation, which was derived in [3] taking the quasi-classical limit of the well-known Lax pair of the KP equation [7, 8].

We remark that, in the dKP reduction  $v = 0$ , the two vector fields are Hamiltonian and the Lax pair (4) takes the form

$$\begin{aligned} \psi_y + p\psi_x - u_x\psi_p &= \psi_y + \{H_1, \psi\}_{(p,x)} = 0, \\ \psi_t + (p^2 + u)\psi_x + (-pu_x + u_y)\psi_p &= \psi_t + \{H_2, \psi\}_{(p,x)} = 0, \end{aligned} \quad (5)$$

in terms of the two Hamiltonians [3]

$$H_1 = \frac{p^2}{2} + u, \quad H_2 = \frac{p^3}{3} + pu - \partial_x^{-1}u_y, \quad (6)$$

where  $\{\cdot, \cdot\}_{(p,x)}$  is the standard Poisson bracket with respect to the canonical variables  $(p, x)$ :

$$\{f, g\}_{(p,x)} \equiv f_p g_x - f_x g_p, \quad (7)$$

leading to the Hamiltonian form of dKP:  $H_{1t} - H_{2y} + \{H_2, H_1\}_{(p,x)} = 0$ .

Since the Lax pair (3) of the dKP-like system (2) is made of vector fields, Hamiltonian in the dKP reduction (1), the eigenfunctions satisfy the following basic properties.

- 1) *The space of eigenfunctions is a ring.* If  $f_1, f_2$  are two solutions of the Lax pair (3), then an arbitrary differentiable function  $F(f_1, f_2)$  of them is a solution of (3).
- 2) *In the dKP reduction  $v = 0$ , the space of eigenfunctions is also a Lie algebra, whose Lie bracket is the natural Poisson bracket (7).* If  $f_1, f_2$  are two solutions of the Lax pair (5), then their Poisson bracket  $\{f_1, f_2\}_{(p,x)}$  is also a solution of (5).

2. Now we consider the Cauchy problem for the dKP system (2) and for the dKP equation (1) within the class of rapidly decreasing real potentials  $u, v$ :

$$u, v \rightarrow 0, \quad (x^2 + y^2) \rightarrow \infty, \quad u \in \mathbb{R}, \quad (x, y) \in \mathbb{R}^2, \quad t > 0, \quad (8)$$

interpreting  $t$  as time and the other two variables  $x, y$  as space variables. To solve such a Cauchy problem by the IST method [9], we construct the IST for the operator  $\hat{L}_1$ , within the class of rapidly decreasing real potentials, interpreting the operator  $\hat{L}_2$  as the time operator.

The localization (8) of the potentials  $u, v$  implies that, if  $f$  is a solution of  $\hat{L}_1 f = 0$ , then

$$\begin{aligned} f(x, y, p) &\rightarrow f_{\pm}(\xi, p), \quad y \rightarrow \pm\infty, \\ \xi &:= x - py; \end{aligned} \quad (9)$$

i.e., asymptotically,  $f$  is an arbitrary function of  $\xi = x - py$  and  $p$ .

A central role in the theory is played by the two real Jost eigenfunctions  $\varphi_{1,2}(x, y, p)$ , the solutions of  $\hat{L}_1 \varphi_{1,2} = 0$  uniquely defined by the asymptotics

$$\varphi_1(x, y, p) \rightarrow \xi, \quad \varphi_2(x, y, p) \rightarrow p, \quad y \rightarrow -\infty. \quad (10)$$

In this paper we often use the compact vector notation:  $\vec{f} = (f_1, f_2)^T$ . Then:

$$\vec{\varphi}(x, y, p) \equiv \begin{pmatrix} \varphi_1(x, y, p) \\ \varphi_2(x, y, p) \end{pmatrix} \rightarrow \begin{pmatrix} \xi \\ p \end{pmatrix} \equiv \vec{\xi}, \quad y \rightarrow -\infty. \quad (11)$$

The Jost eigenfunction  $\vec{\varphi}$  is the solution of the linear integral equations  $\vec{\varphi} = \vec{\xi} + \hat{G}(-v_x \vec{\varphi}_x + u_x \vec{\varphi}_p)$ , for the Green's function  $G(x, y, p) = \theta(y) \delta(x - py)$ .

The  $y = +\infty$  limit of  $\vec{\varphi}$  defines the natural scattering vector  $\vec{\sigma}$  for  $\hat{L}_1$ :

$$\lim_{y \rightarrow +\infty} \vec{\varphi}(x, y, p) \equiv \vec{\mathcal{S}}(\vec{\xi}) = \vec{\xi} + \vec{\sigma}(\vec{\xi}). \quad (12)$$

The direct problem is the transformation from the real potentials  $u, v$ , functions of the two real variables  $(x, y)$ , to the two real scattering data  $\sigma_1, \sigma_2$ , the components of the scattering vector  $\vec{\sigma}$ , functions of the two real variables  $(\xi, p)$ . Therefore the mapping is consistent. The impact of the dKP reduction  $v = 0$  on these and other data will be shown below.

A crucial role in the IST theory for the vector field  $\hat{L}_1$  is also played by the analytic eigenfunctions  $\vec{\psi}_{\pm}(x, y, p)$ , the solutions of  $\hat{L}_1 \vec{\psi}_{\pm} = \vec{0}$  satisfying the integral equations

$$\begin{aligned} \vec{\psi}_{\pm}(x, y, p) &= \int_{\mathbb{R}^2} dx' dy' G_{\pm}(x - x', y - y', p) [-v_{x'}(x', y') \vec{\psi}_{\pm x'}(x', y', p) + \\ &u_{x'}(x', y') \vec{\psi}_{\pm p}(x', y', p)] + \vec{\xi}, \end{aligned} \quad (13)$$

where  $G_{\pm}$  are the analytic Green's functions

$$G_{\pm}(x, y, p) = \pm \frac{1}{2\pi i [x - (p \pm i\epsilon)y]}. \quad (14)$$

The analyticity properties of  $G_{\pm}(x, y, p)$  in the complex  $p$  - plane imply that  $\vec{\psi}_+(x, y, p)$  and  $\vec{\psi}_-(x, y, p)$  are analytic, respectively, in the upper and lower halves of the  $p$  - plane, with the following asymptotics, for large  $p$ :

$$\begin{aligned} \vec{\psi}_{\pm}(x, y, p) &= \vec{\xi} + \frac{1}{p} \vec{U}(x, y) + \vec{O}\left(\frac{1}{p^2}\right), \quad |p| \gg 1, \\ \vec{U}(x, y) &\equiv \begin{pmatrix} -yu(x, y) - v(x, y) \\ u(x, y) \end{pmatrix}. \end{aligned} \quad (15)$$

It is important to remark that the analytic Green's functions (14) exhibit the following asymptotics for  $y \rightarrow \pm\infty$ :

$$\begin{aligned} G_{\pm}(x - x', y - y', p) &\rightarrow \pm \frac{1}{2\pi i [\xi - \xi' \mp i\epsilon]}, \quad y \rightarrow +\infty, \\ G_{\pm}(x - x', y - y', p) &\rightarrow \pm \frac{1}{2\pi i [\xi - \xi' \pm i\epsilon]}, \quad y \rightarrow -\infty, \end{aligned} \quad (16)$$

entailing that *the  $y = +\infty$  asymptotics of  $\vec{\psi}_+$  and  $\vec{\psi}_-$  are analytic respectively in the lower and upper halves of the complex plane  $\xi$ , while the  $y = -\infty$  asymptotics of  $\vec{\psi}_+$  and  $\vec{\psi}_-$  are analytic respectively in the upper and lower halves of the complex plane  $\xi$*  (similar features have been observed first in [10] and later in [6]).

The Jost eigenfunctions  $\varphi_{1,2}$  form a basis; thus any solution  $f$  of  $\hat{L}_1 f = 0$  is a function of  $\vec{\varphi}$ . The analytic eigenfunctions  $\vec{\psi}_{\pm}$  possess the representations:

$$\vec{\psi}_{\pm} = \vec{\mathcal{K}}_{\pm}(\vec{\varphi}) = \vec{\varphi} + \vec{\chi}_{\pm}(\vec{\varphi}), \quad (17)$$

defining the spectral data  $\vec{\chi}_{\pm}$ .

Since the  $y \rightarrow -\infty$  limit of (17) reads:

$$\lim_{y \rightarrow -\infty} \vec{\psi}_{\pm} - \vec{\xi} = \vec{\chi}_{\pm}(\vec{\xi}), \quad (18)$$

the above analyticity properties of the LHS of (18) in the complex  $\xi$  - plane imply that  $\vec{\chi}_+(\vec{\xi})$  and  $\vec{\chi}_-(\vec{\xi})$  are analytic respectively in the upper and lower halves of the complex plane  $\xi$ , decaying at  $\xi \sim \infty$  like  $O(\xi^{-1})$ . Therefore their Fourier transforms  $\tilde{\chi}_+(\vec{\omega})$  and  $\tilde{\chi}_-(\vec{\omega})$  have support respectively on the positive and negative  $\omega_1$  semi-axes.

The spectral vectors  $\vec{\chi}_\pm$  can be constructed from the scattering vector  $\vec{\sigma}$  through the following linear integral equations

$$\begin{aligned}\tilde{\chi}_+(\vec{\omega}) + \theta(\omega_1) \left( \tilde{\vec{\sigma}}(\vec{\omega}) + \int_{\mathbb{R}^2} d\vec{\eta} \tilde{\chi}_+(\vec{\eta}) Q(\vec{\eta}, \vec{\omega}) \right) &= \vec{0}, \\ \tilde{\chi}_-(\vec{\omega}) + \theta(-\omega_1) \left( \tilde{\vec{\sigma}}(\vec{\omega}) + \int_{\mathbb{R}^2} d\vec{\eta} \tilde{\chi}_-(\vec{\eta}) Q(\vec{\eta}, \vec{\omega}) \right) &= \vec{0},\end{aligned}\tag{19}$$

involving the Fourier transforms  $\tilde{\vec{\sigma}}$  and  $\tilde{\chi}_\pm$  of  $\vec{\sigma}$  and  $\vec{\chi}_\pm$ :

$$\tilde{\vec{\sigma}}(\vec{\omega}) = \int_{\mathbb{R}^2} d\vec{\xi} \vec{\sigma}(\vec{\xi}) e^{-i\vec{\omega} \cdot \vec{\xi}}, \quad \tilde{\chi}_\pm(\vec{\omega}) = \int_{\mathbb{R}^2} d\vec{\xi} \vec{\chi}_\pm(\vec{\xi}) e^{-i\vec{\omega} \cdot \vec{\xi}}\tag{20}$$

and the kernel:

$$Q(\vec{\eta}, \vec{\omega}) = \int_{\mathbb{R}^2} \frac{d\vec{\xi}}{(2\pi)^2} e^{i(\vec{\eta} - \vec{\omega}) \cdot \vec{\xi}} [e^{i\vec{\eta} \cdot \vec{\sigma}(\vec{\xi})} - 1].\tag{21}$$

To prove this result, one first evaluates (17) at  $y = +\infty$ , obtaining

$$\left( \lim_{y \rightarrow \infty} \vec{\psi}_\pm - \vec{\xi} \right) = \vec{\sigma}(\vec{\xi}) + \vec{\chi}_\pm(\vec{\xi} + \vec{\sigma}(\vec{\xi})).\tag{22}$$

Applying the integral operator  $\int_{\mathbb{R}^2} d\vec{\xi} e^{-i\vec{\omega} \cdot \vec{\xi}}$  for  $\omega_1 > 0$  and  $\omega_1 < 0$  respectively to equations (22)<sub>+</sub> and (22)<sub>-</sub>, using the above analyticity properties and the Fourier representations of  $\vec{\chi}_\pm$  and  $\vec{\sigma}$ , one obtains equations (19).

The reality of the potentials:  $u, v \in \mathbb{R}$  implies that, for  $p \in \mathbb{R}$ ,  $\overline{\vec{\varphi}} = \vec{\varphi}$ ,  $\overline{\vec{\psi}_+} = \vec{\psi}_-$ ; consequently:  $\overline{\vec{\sigma}} = \vec{\sigma}$ ,  $\overline{\vec{\chi}_+} = \vec{\chi}_-$ .

3. An inverse problem can be constructed from equations (17). Subtracting  $\vec{\xi}$  from equations (17)<sub>-</sub> and (17)<sub>+</sub>, applying respectively the analyticity projectors  $\hat{P}_+$  and  $\hat{P}_-$ :

$$\hat{P}_\pm \equiv \pm \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{dp'}{p' - (p \pm i\epsilon)}.\tag{23}$$

and adding up the resulting equations, one obtains the following nonlinear integral equation for the Jost eigenfunction  $\vec{\varphi}$ :

$$\begin{aligned}\vec{\varphi}(x, y, p) + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{dp'}{p' - (p + i\epsilon)} \vec{\chi}_-(\vec{\varphi}(x, y, p')) - \\ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{dp'}{p' - (p - i\epsilon)} \vec{\chi}_+(\vec{\varphi}(x, y, p')) = \vec{\xi}.\end{aligned}\tag{24}$$

Once  $\vec{\varphi}$  is reconstructed from (24), the analytic eigenfunctions follow from (17), and  $u, v$  from equation (15). This inversion procedure was first introduced in [11] and also used in [6].

4. As  $u, v$  evolve in time according to (2), the  $t$ -dependence of the spectral data  $\vec{\mathcal{S}}$  and  $\vec{\mathcal{K}}_{\pm}$ , defined in (12) and (17), is described by the equations:

$$\begin{aligned}\Sigma_1(\xi, p, t) &= t(\Sigma_2(\xi - p^2t, p, 0))^2 + \Sigma_1(\xi - p^2t, p, 0), \\ \Sigma_2(\xi, p, t) &= \Sigma_2(\xi - p^2t, p, 0),\end{aligned}\tag{25}$$

where  $\Sigma_1$  and  $\Sigma_2$  are the two components of the vector  $\vec{\Sigma}$ , identifiable with each of the spectral vectors  $\vec{\mathcal{S}}$  and  $\vec{\mathcal{K}}_{\pm}$ . To prove it, we first observe that

$$\begin{aligned}\phi_1(x, y, t, p) &\equiv \varphi_1(x, y, t, p) - t\varphi_2^2(x, y, t, p), \\ \phi_2(x, y, t, p) &\equiv \varphi_2(x, y, t, p)\end{aligned}\tag{26}$$

are a basis of common Jost eigenfunctions of  $\hat{L}_1$  and  $\hat{L}_2$ . The  $y = +\infty$  limit of equation  $\hat{L}_2\phi_2 = 0$  yields  $\mathcal{S}_{2t} + p^2\mathcal{S}_{2\xi} = 0$ , while the  $y = +\infty$  limit of equation  $\hat{L}_2\phi_1 = 0$  yields  $(\partial_t + p^2\partial_{\xi})(\mathcal{S}_1 - t\mathcal{S}_2^2) = 0$ , whose solutions are (25) for  $\vec{\mathcal{S}}$ . Analogously,

$$\begin{aligned}\pi_{\pm 1}(x, y, t, p) &\equiv \psi_{\pm 1}(x, y, t, p) - t\psi_{\pm 2}^2(x, y, t, p), \\ \pi_{\pm 2}(x, y, t, p) &\equiv \psi_{\pm 2}(x, y, t, p)\end{aligned}\tag{27}$$

are a basis of common analytic eigenfunctions of  $\hat{L}_1$  and  $\hat{L}_2$ ; therefore

$$\pi_{\pm 1} = \check{\mathcal{K}}_{\pm 1}(\phi_1, \phi_2), \quad \pi_{\pm 2} = \check{\mathcal{K}}_{\pm 2}(\phi_1, \phi_2),\tag{28}$$

for some functions  $\check{\mathcal{K}}_{\pm 1,2}$  depending on  $x, y, t, p$  only through  $\vec{\phi}$ . Comparing at  $t = 0$  these equations with equations (17), one expresses  $\check{\mathcal{K}}_{\pm 1,2}$  in terms of  $\mathcal{K}_{\pm 1,2}$ , obtaining equations (25) for  $\mathcal{K}_{\pm 1,2}$ .

We observe the unusual resonant character of the explicit  $t$ -dependence (25) of the spectral data, if compared to the more elementary one, obtained in [6], for the heavenly equation [12].

5. In the Hamiltonian dKP reduction  $v = 0$ , the transformations  $\vec{\xi} \rightarrow \vec{\mathcal{S}}(\vec{\xi})$ ,  $\vec{\xi} \rightarrow \vec{\mathcal{K}}_{\pm}(\vec{\xi})$  are constrained to be canonical:

$$\{\mathcal{S}_1, \mathcal{S}_2\}_{\vec{\xi}} = \{\mathcal{K}_{\pm 1}, \mathcal{K}_{\pm 2}\}_{\vec{\xi}} = 1.\tag{29}$$

To prove it, we observe that the Poisson bracket of the eigenfunctions  $\varphi_1$  and  $\varphi_2$  is also an eigenfunction:  $\varphi_3 \equiv \{\varphi_1, \varphi_2\}_{(x,p)}$ ,  $\hat{L}_1\varphi_3 = 0$ . Using

the asymptotics (10), one infers that  $\varphi_3 \rightarrow 1$ , at  $y \rightarrow -\infty$ ; therefore, by uniqueness,  $\varphi_3 = 1$ . Evaluating now the Poisson bracket  $\varphi_3$  at  $y = +\infty$  and using (12), one obtains the constraint (29) for  $\vec{\mathcal{S}}$ . We also observe that the eigenfunctions  $\{\psi_{+1}, \psi_{+2}\}_{(x,p)}$  and  $\{\psi_{-1}, \psi_{-2}\}_{(x,p)}$  are analytic in the upper and lower  $p$  plane and go to 1 at  $|p| \rightarrow \infty$ . Since 1 is also an eigenfunction, by uniqueness they are identically 1:  $\{\psi_{\pm 1}, \psi_{\pm 2}\}_{(x,p)} = 1$ . Therefore, from the equations:

$$\{\psi_{\pm 1}, \psi_{\pm 2}\}_{(x,p)} = \{\mathcal{K}_{\pm 1}, \mathcal{K}_{\pm 2}\}_{(\varphi_1, \varphi_2)} \{\varphi_1, \varphi_2\}_{(x,p)} = 1, \quad (30)$$

consequence of (17), one infers the constraints (29) for  $\vec{\mathcal{K}}_{\pm}$ .

6. It is well-known (see, f.i., [13]) that linear first order PDEs like (3),(4) are intimately related to systems of ordinary differential equations describing their characteristic curves. The Hamiltonian dynamical systems associated with the vector fields  $\hat{L}_{1,2}$  of dKP are:

$$\hat{L}_1 : \begin{cases} \frac{dx}{dy} = p = \{H_1, x\}_{(p,x)}, \\ \frac{dp}{dy} = -u_x = \{H_1, p\}_{(p,x)} \end{cases} \quad (31)$$

$$\hat{L}_2 : \begin{cases} \frac{dx}{dt} = p^2 + u, = \{H_2, x\}_{(p,x)}, \\ \frac{dp}{dt} = -pu_x + u_y = \{H_2, p\}_{(p,x)}, \end{cases} \quad (32)$$

Therefore the dKP equation characterizes the class of time - dependent potentials for which the Newtonian flow (31) commutes with a flow with cubic, in the momentum  $p$ , Hamiltonian.

There is also a deep connection between the above IST and the  $y$ -scattering theory for the commuting flows (31) and (32). Let  $\vec{\phi}(x, y, t, p)$  be the basis of common eigenfunctions of  $\hat{L}_1$  and  $\hat{L}_2$  defined in (26); then, solving the system  $\vec{\omega} = \vec{\phi}(x, y, t, p)$  with respect to  $x$  and  $p$  (assuming local invertibility), one obtains the following common solution of (31) and (32):

$$\vec{\omega} = \vec{\phi}(x, y, t, p) \Leftrightarrow \begin{pmatrix} x \\ p \end{pmatrix} = \vec{r}(y, t; \vec{\omega}) \sim \begin{pmatrix} \omega_2 y + \omega_2^2 t + \omega_1 \\ \omega_2 \end{pmatrix}, \quad y \sim -\infty. \quad (33)$$

The  $y = +\infty$  limit of the solution  $\vec{r}(y, t; \vec{\omega})$ :

$$\begin{pmatrix} x \\ p \end{pmatrix} \sim \begin{pmatrix} \Omega_2(\vec{\omega})y + \Omega_2^2(\vec{\omega})t + \Omega_1(\vec{\omega}) \\ \Omega_2(\vec{\omega}) \end{pmatrix}, \quad y \sim +\infty \quad (34)$$

defines the scattering vector  $\vec{\Delta}(\vec{\omega}) = \vec{\Omega}(\vec{\omega}) - \vec{\omega}$  of (31) and (32), which is connected to the IST data  $\vec{\mathcal{S}}$  by inverting the system  $\vec{\omega} = \vec{\mathcal{S}}(x - py - p^2t, p, 0)$  with respect to  $x$  and  $p$ :

$$\vec{\omega} = \vec{\mathcal{S}}(x - py - p^2t, p, 0) \Leftrightarrow \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} \Omega_2(\vec{\omega})y + \Omega_2^2(\vec{\omega})t + \Omega_1(\vec{\omega}) \\ \Omega_2(\vec{\omega}) \end{pmatrix}. \quad (35)$$

The transformation  $\vec{\omega} \rightarrow \vec{\Omega}(\vec{\omega})$  is clearly canonical:  $\{\vec{\Omega}_1, \vec{\Omega}_2\}_{(\omega_1, \omega_2)} = 1$ .

Since the dynamical system (31) describes the motion of a Newtonian particle in the plane subjected to a generic time - dependent potential  $u(x, y)$ , as a byproduct of the IST of this paper one can reconstruct, from the scattering vector  $\vec{\Delta}(\vec{\omega})$  of the dynamical system (31), the time dependent potential  $u$ .

*Remark 1.* There are two other ways to do the inverse problem. The first one is the linear version of the nonlinear problem (24), obtained *exponentiating the Jost and analytic eigenfunctions* used so far. Consider the following scalar functions:

$$\Phi(x, y, p; \vec{\alpha}) \equiv e^{i\vec{\alpha} \cdot \vec{\varphi}(x, y, p)}, \quad \Psi_{\pm}(x, y, p; \vec{\alpha}) \equiv e^{i\vec{\alpha} \cdot \vec{\psi}_{\pm}(x, y, p)}, \quad \vec{\alpha} \in \mathbb{R}^2. \quad (36)$$

Due to the ring property of the space of eigenfunctions, also  $\Phi(x, y, p; \vec{\alpha})$  and  $\Psi_{\pm}(x, y, p; \vec{\alpha})$  are eigenfunctions;  $\Phi(x, y, p; \vec{\alpha})$  is characterized by the asymptotics  $\Phi \rightarrow \exp(i\vec{\alpha} \cdot \vec{\xi})$ ,  $y \rightarrow -\infty$ , while  $\Psi_{\pm}(x, y, p; \vec{\alpha})$  are analytic respectively in the upper and lower halves of the  $p$  plane, with asymptotics:  $\Psi_{\pm} = \exp(i\vec{\alpha} \cdot \vec{\xi})[1 + p^{-1}\vec{\alpha} \cdot \vec{U}(x, y) + O(p^{-2})]$ .

Exponentiating the representations (17), one obtains the expansions of the analytic eigenfunctions  $\Psi_{\pm}$  in terms of the Jost eigenfunction  $\Phi$ :

$$\begin{aligned} \Psi_{\pm}(x, y, p; \vec{\alpha}) &= \Phi(x, y, p; \vec{\alpha}) + \int_{\mathbb{R}^2} d\vec{\beta} K_{\pm}(\vec{\alpha}, \vec{\beta}) \Phi(x, y, p; \vec{\beta}), \\ K_{\pm}(\vec{\alpha}, \vec{\beta}) &\equiv \int_{\mathbb{R}^2} \frac{d\vec{\xi}}{(2\pi)^2} e^{i(\vec{\alpha} - \vec{\beta}) \cdot \vec{\xi}} [e^{i\vec{\alpha} \cdot \vec{\chi}_{\pm}(\vec{\xi})} - 1]. \end{aligned} \quad (37)$$

Multiplying the equations (37)<sub>+</sub> and (37)<sub>-</sub> by  $\exp(-i\vec{\alpha} \cdot \vec{\xi})$ , subtracting 1, applying respectively  $\hat{P}_-$  and  $\hat{P}_+$ , and adding the resulting equations, one obtains the following *linear integral equation* for  $\Phi$ :

$$\begin{aligned} \Phi(p; \vec{\alpha}) + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{dp'}{p' - (p + i\epsilon)} \int_{\mathbb{R}^2} d\vec{\beta} K_{-}(\vec{\alpha}, \vec{\beta}) \Phi(p'; \vec{\beta}) e^{i\vec{\alpha} \cdot (\vec{\xi}(p) - \vec{\xi}(p'))} - \\ - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{dp'}{p' - (p - i\epsilon)} \int_{\mathbb{R}^2} d\vec{\beta} K_{+}(\vec{\alpha}, \vec{\beta}) \Phi(p'; \vec{\beta}) e^{i\vec{\alpha} \cdot (\vec{\xi}(p) - \vec{\xi}(p'))} = e^{i\vec{\alpha} \cdot \vec{\xi}(p)}, \end{aligned} \quad (38)$$



in which we have omitted, for simplicity, the parametric dependence on  $(x, y)$ . Once  $\Phi$  is reconstructed from (38) and, via (37),  $\Psi_{\pm}$  are also known, the potentials are reconstructed in the usual way from the asymptotics of  $\Psi_{\pm}$ .

The third version of the inverse problem is a more traditional (nonlinear) Riemann-Hilbert (RH) problem. Solving the algebraic system  $(17)_{-}$  with respect to  $\vec{\varphi}$ :  $\vec{\varphi} = L(\vec{\psi}_{-})$  (assuming local invertibility) and replacing this expression in the algebraic system  $(17)_{+}$ , one obtains the representation of the analytic eigenfunction  $\vec{\psi}_{+}$  in terms of the analytic eigenfunction  $\vec{\psi}_{-}$ :

$$\vec{\psi}_{+} = \vec{\mathcal{R}}(\vec{\psi}_{-}) = \vec{\psi}_{-} + \vec{R}(\vec{\psi}_{-}), \quad p \in \mathbb{R}, \quad (39)$$

which defines a *vector nonlinear RH problem on the real  $p$  axis*. The RH data  $\vec{\mathcal{R}}$  are therefore constructed from the data  $\vec{\mathcal{K}}$  by algebraic manipulation. Viceversa, given the RH data  $\vec{R}$ , one constructs the solutions  $\vec{\psi}_{\pm}$  of the nonlinear RH problem (39) and, via the asymptotics (15), the potentials.

As for the other spectral data, one can show that the  $t$ -dependence of  $\vec{\mathcal{R}}$  is described by (25) and the dKP constraint reads  $\{\mathcal{R}_1, \mathcal{R}_2\}_{(\xi, p)} = 1$ , while the reality constraint takes the form:  $\vec{\mathcal{R}}(\overline{\vec{\mathcal{R}}(\xi, \lambda)}, \lambda) = \vec{\xi}$ ,  $\forall \vec{\xi}$ , for  $p \in \mathbb{R}$ .

*Remark 2.* Dressing schemes can be formulated from the three different inverse problems presented in this paper in a straightforward way.

*Remark 3.* The IST constructed in this paper allows one to solve the Cauchy problem for the whole hierarchy of PDEs arising from the commutativity equation  $[\hat{L}_1, \hat{L}_2^{(n)}] = 0$ , where the coefficients of the vector field  $\hat{L}_2^{(n)}$  are polynomials in  $p$  of arbitrary degree  $n \in \mathbb{N}$ .

*Remark 4.* There are deep similarities between the Cauchy problem for dKP and the Cauchy problem for the heavenly equation, recently solved in [6], since they are both based on the IST for Hamiltonian vector fields (the dKP equation is actually a geometric reduction of the heavenly equation [3]). There is, however, an important difference between these two cases. The vector fields of the dKP equation contain partial derivatives with respect to the spectral parameter  $p$ , unlike the case of the heavenly equation [6].

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